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The Improvement of System Reliability Optimization Model and Finding an Optimal Solution

Seyed-Jafar Sadjadi
Department of Industrial Engineering,
Iran University of Science and Technology, Tehran, Iran
sjsadjadi@iust.ac.ir
(Corresponding Author)

Saeed Jafari
Department of Industrial Engineering,
Iran University of Science and Technology, Tehran, Iran

Abstract. When a production facility is designed, there are various parameters affecting the number machines such as production capacity and reliability. It is often a tedious task to optimize different objectives simultaneously. The other issue is the uncertainty in many design parameters which makes it difficult to reach a desirable solution. In this paper, we present a new mathematical model with two objectives. The primary objective function is considered to be the production capacity and the secondary objective function is total reliability. The proposed model is formulated on different units of production which are connected together in serial form and for each unit, we may have various machines. The resulted model is formulated using recent advances of robust optimization and solution procedure is analyzed with some numerical examples.

Keywords: Reliability Series-Parallel System, Multi-Objective Optimization, Two-Constraint Model, Nonlinear programming.

1. Introduction

Each system may include subsystems or entities in several ways. For example, a system can be series, parallel, or combination of series and parallel consisting of subsystem. Naturally, system reliability optimization depends directly on the type of relationship in subsystems of the system. In this paper, we aim to optimize the reliability of a series-parallel system that has more than one constraint. Given that the reliability of a series - parallel system is non-linear function and usually the existing systems and the real issues have more than a constraint, obtaining optimum solution for this type of very complex problems and large-sized problems is almost impossible. Most previous studies have focused on solving a constraint model. Therefore, general solution has not been obtained for the problem of optimizing reliability with more than one constraint.

In recent years, papers have been published which are called robust optimization. Molavi et al (1995) provide a solution that integrates the goal programming formulation with scenario-based difficult data description. Sooyster (1973) offers the linear optimization model to provide a solution. In the past, heuristic methods were presented to get optimum solution for reliability optimization issue of series- parallel system. However, most of these solutions can be used only for small sized problems and the methods used previously are used for specific problems. In this paper, we will model the reliability of a series - parallel system and will obtain optimal solution for problem reliability. So, we will obtain the maximum reliability considering space, human power, and power consumption constraints.

In this paper, we will use the multi-objective optimization method for obtaining optimum solution. Therefore first, we will maximize the production capacity by considering the existing constraints. Then, according to the maximum response obtained in the first phase, we will obtain a maximum solution for the target function of reliability.

2. Modeling the First Phase of the Problem

In the first section, we aim to obtain optimal solution for production capacity maximization in a parallel-series system. In other words, we want to maximize the minimum production capacity by considering the linear constraints such as cost, space, power consumption rate of each machine, and etc.

The model of above-mentioned problem is finally written as follows by considering the constraints, the upper and lower bound limitation, and positive variables:

$$
\begin{aligned}
&\max \quad \min \quad \{p_i x_i\} \\
&subject \quad to \quad Ax \leq b \\
&\qquad\qquad\qquad l \leq x \leq u \\
&\qquad\qquad\qquad x_i \geq 1, 2, \ldots
\end{aligned}
\tag{1}
$$

By applying simple changes, the above max-min problem may be transformed to a maximization problem as follows:

$$
\begin{aligned}
&\max \quad w \\
&subject \quad to \quad p_i x_i \geq w \\
&\qquad\qquad\qquad Ax \leq b \\
&\qquad\qquad\qquad l \leq x \leq u \\
&\qquad\qquad\qquad x_i \geq 1, 2, \ldots \\
&\qquad\qquad\qquad w \geq 0
\end{aligned}
\tag{2}
$$

Now, we want to obtain the optimal solution of the above problem in a state that part of the above mentioned coefficients do not exist definitely.

In this section, we provide a general framework and the formulation of discrete optimization problems.

2.1. Robust Formulation of Discrete Optimization Problem

We assume the U, L, C as n-vector and the matrix m×n and b as the m-vector. We consider the following MIP (Mixed Integer Programming) in a set of n variables:

$$\begin{aligned} \text{minimize} \quad & c^t x \\ \textit{subject to} \quad & Ax \le b \\ & l \le x \le u \\ & x_i \in Z, \quad i = 1, \dots, k, \end{aligned} \tag{3}$$

Without missing the generality of problem, we assume that the data uncertainty effects only impacts on the matrix elements of C, A, and not the vector b. In this case, we present new variable x_{n+1} and write:

$$AX - bx_{n+1} \le 0,\ L \le x \le u,\ 1 \le x_{n+1} \le 1$$

Therefore, A is more than B.

Uncertainty Model U

a) Uncertainty of matrix A: Γ_0

Put $N = \{1, 2, \ldots, n\}$. Each entry of $a_{ij}, j \in \mathrm{N}$ and the independent, symmetric, and restricted random is modeled as $\hat{a}_{ij}, j \in \mathrm{N}$ But, it has unknown distribution that these values is given to it:

$$\left[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}\right]$$

b) Uncertain factors in cost vector C: each C_j and J϶N entry has $\left[C_j, C_j + d_j\right]$ values. This is while the D_j represents the deviation from the nominal rate of cost C_j.

It should be noted that the probable case is allowed to be $a_{ij} = 0$ or $d_i = 0$. Also, it is noteworthy that the only assumption of coefficients distribution of a_{ij} is that they are symmetrical.

2.2. Robust MIP Formulation

For robust targets and for every i, we introduce the $\Gamma_i (i = 0, 1, \ldots, m)$ that takes value in the range of $[0, |J_i|]$ and it is $\Gamma_i = \{j / \hat{a}_{ij} > 0\}$. We assume the Γ_0 as an integer. While the $\Gamma_i (i = 1, 2, \ldots, m)$ are not necessarily integers.

The role of parameter Γ_i in above-mentioned constraints is adapting the robust of proposed method against the solution conservative level. Consider the i th constraint of nominal issue $a_i' x \le bi$

Consider the J_i as a set of coefficients $a_{ij} i \in J_i$ that is dependent on uncertain parameter. In other words, $a_{ij} i \in J_i$ independently takes values according to the symmetrical distribution with exact equality with nominal value a_{ij} in the range of $[a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$. We strictly state that we cannot say if all $a_{ij} i \in J_i$, they will change. We aim to avoid all things that Γ_i of these coefficients are allowed to change and a_{it} coefficient with the highest $(\Gamma_i - [\Gamma_i])\hat{a}_{it}$ changes. In other words, we point out that natural numbers will have limited behaviors; so that, only the subset of coefficients will have reverse effect on the solution. We then ensure that if the natural numbers act like this, the robust solution will greatly simplified. Also, we show that necessarily the distributions that we allow them will be symmetrical. Even if the changes will be more than Γ_i, then the robust solution will be feasible with high probability. We call the Γ_i as protection level for the i th constraint.

The parameter Γ_0 controls the level of accuracy in the target. We are interested to find an optimal solution that is optimal under some Γ_0 cost weights. In general, the highest amount of Γ_0 increases the accuracy rate in the highest nominal expense. The detailed example is proposed for problem (3) is as follows.

$$\text{minimize} \quad c'x + \max_{\{S_0 | S_0 \subseteq J_0, |S_0| \leq \Gamma_0\}} \left\{ \sum_{j \in S_0} d_j \mid x_j \mid \right\} \tag{4}$$

$$s.t.$$

$$\max \quad t$$

$$\sum_j a_{ij} x_j + \left\{ S_i U \{t_i\} \middle| S_i \subseteq J_i \middle| S_i \; / \leq \left[\Gamma_i\right], t_i \in J_i \setminus S_i \right\} \leq b_i, \; \forall i$$

$$t = \left\{ \sum_{j \in S_i} \hat{a}_{ij} \left| x_j \right| + \left(\Gamma_i - \left[\Gamma_i \right] \right) \hat{a}_{it_i} \left| x_{ti} \right| \right\}$$

$$l \leq x \leq u$$

$$x_i \in Z, \qquad \forall\, i = 1, \dots, k.$$

Then we show that the solution proposed by Sim and Britsimas (2001) is drived from linear optimization to discrete optimization.

Theory: the problem (4) is equivalent to following MIP formulation: (Britsimas - Sim -2002).

$$
\begin{aligned}
&\text{minimize} \quad c'x + z_0\Gamma_0 + \sum\nolimits_{j\in J_0} P_{0j} \\
&\sum_j a_{ij}x_j + z_i\Gamma_i + \sum_{j\in J_i} P_{ij} \le b_i \quad \forall_i \\
&z_0 + P_{0j} \ge d_j y_j \qquad \forall j\in J_0 \\
&z_i + P_{ij} \ge \hat{a}_{ij} y_j \qquad \forall i\ne 0, j\in J_i \\
&P_{ij} \ge 0 \qquad \forall i, j\in J_i \\
&y_i \ge 0 \qquad \forall i \\
&z_i \ge 0 \qquad \forall i \\
&-y_j \le x_j \le y_j \qquad \forall j \\
&l_j \le x_j \le u_j \qquad \forall j \\
&x_i \in Z \qquad i=1,\dots,k.
\end{aligned}
\tag{5}
$$

If we consider the coefficients of other constraints- other than the constraint coefficient of production capacity- that is matrix A, the problem will be a little more complicated than the situation which we consider only one constraint as uncertain.

In this section, all constraint coefficients are considered as uncertain and the offered problem for maximization of production capacity is rewritten accurately and robustly as follows:

$$
\begin{aligned}
&\max \quad w \\
&w - p_i x_i + z_i\Gamma_i + P_i \le 0 \\
&z_i + P_i \ge p_i y_i \qquad \forall j\in J_0 \\
&\sum_j a_{ij}x_j + z_i\Gamma_i + \sum_{j\in J_i} P_{ij} \le b_i \qquad \forall i \\
&z_i + P_{ij} \ge \hat{a}_{ij} y_j \qquad \forall i\ne 0,\ j\in J_i \\
&P_i \ge 0 \qquad \forall i,\ j\in J_i \\
&y_i \ge 0 \qquad \forall i \\
&z_i \ge 0 \qquad \forall i \\
&w \ge 0 \qquad \forall j \\
&-y_j \le x_j \le y_j \qquad \forall j \\
&l_i \le x_i \le u_i \qquad \forall i \\
&x_i \ge 1, 2, \dots \qquad i = 1, \dots, k.
\end{aligned}
\tag{6}
$$

3. The Second Function of Optimization

In this section, we evaluate our proposed heuristic solution. The complexity of parallel-series problem is np-hard type. Therefore, we expect that optimal solution not be achieved for large-scale problems. However, for a simple model in which there is only one constraint, optimal solution may be found.

1. The reliability of series- parallel system is equal to:

$$\max \sum_{i=1}^{s} \ln[1-(1-R_i)^{x_i}] \tag{7}$$

Where we have: $q = 1 - R$.

2. If the price of a car will be in the class C_i, *I,* the total price of class i will be equal to $c_i x_i$. Where, xi is the number of cars in a class as parallel.

3. The total budget is allocated for the project is assumed to be C.

4. The problem of determining the optimal plan of establishment is that the total cost should not be more than the determined budget.

In this case, the problem is formulated as follows:

$$\max\ z = [1-(1-R_1)^{n_1}] \times [1-(1-R_2)^{n_1}] \ldots [\ 1-(1-R_s)^{n_s}] \tag{8}$$

$$s.t. \sum_{i=1}^{s} c_i n_i \leq C$$

In this model, the number of cars in each class related the xi to the reliability and cost.

However, this model has one constraint. If the constraints are increased, the space, safety level, the amount of used energy, and etc constraints may be entered to the model.

However, if the logarithm of the target function are maximized, the main function will be maximized too. Therefore, the target function can be considered as follows. (Arianezhad and Sajjadi, 2001).

$$\max \sum_{i=1}^{s} \ln[1-(1-R_i)^{x_i}] \tag{9}$$

The first step to find the optimal set is calculating the following equation. Suppose, the Δ_i represents the increase in the logarithm because of investing one more money unit on i th car and it is defined as follows.

$$\Delta_i = \frac{\ln\left[1-(1-R_i)^{n_i+1}\right]-\ln\left[1-(1-R_i)^{n_i}\right]}{c_i} \tag{10}$$

The steps of algorithm:

1. Set the ni (the number of cars in each class) equal to 1 and set the value of cost variable equal to zero.
2. Calculate the value ofΔ_i for all i_s
3. Calculate the maximum value of Δ_i and call it Δ_k.
4. Calculate the cost value according to this formula: cost = cost + ci.
5. If it is cost <c, add 1 to n $_k$ and re-calculate Δ_k and go to the third step. Otherwise, the algorithm ends.

This method calculates Δ_i for each subsystem, selects one case, and increases it one unit. In fact, Δ_i calculates the relative change in the overall reliability.

This process is continued until the available funds for the system to be identified. However, this process cannot be used for all reliability optimization problems; because, it is probable that there will be more than one constraint.

The following code shows the implementation of the proposed method:

In the following code, c shows the limit of cost and C is the coefficient matrix of each device's cost.

The above method is proposed as an approach for finding a heuristic solution. Note that if there exists only one constraint, then the final solution will be optimal.

The calculation order for this method is as o (l.m.s). This method is used for small values of L, M, or almost high speed S in the computer. If the value of inputted parameters will be larger than 15, the optimal solution of the above model will not available for practical problems.

```
Begin
  Set R  ,A,b,n=[1,...,1];
      While CX ≤ c
        Temp ← 0;
            for I = 1 to s
```

$$\Delta_i = \frac{\ln\left[1-(1-R_i)^{n_i+1}\right] - \ln\left[1-(1-R_i)^{n_i}\right]}{c_i}$$

```
                if temp ≤ Δi                                  (11)
              temp = Δi;
             index = i
            end if
          end for
        n(index) = n(index) + 1;
    end while
End
```

Therefore, we should have an upper bound to calculate the results of our proposed method. We may use the above method if we consider only one constraint. Thereby, our guess for the optimal solution will be good.

4. Case Study

4.1. Modeling

A factory has production line with 5 machines for manufacturing transformers. Several similar machines are added to the system in parallel to improve the reliability of the whole system and to reduce the damages and failures in production line. Finally, after collecting the above data and modeling the problem, the following model is obtained to optimize the production capacity of system:

$$\begin{aligned} \text{max- min} \quad & 20x_1 + 10x_2 + 50x_3 + 10x_4 + 30x_5 \\ s.t. \quad & 2x_1 + 1x_2 + 1x_3 + 4x_4 + 3x_5 \le 30 \\ & 2x_1 + 1x_2 + 4x_3 + 2x_4 + 6x_5 \le 40 \\ & 3x_1 + 2x_2 + 5x_3 + 7x_4 + 3x_5 \le 70 \end{aligned} \tag{12}$$

4.2 Solving the Model and Analyzing the Answer

If we do not consider the uncertainty condition for the problem (i.e. $\Gamma_i = 0$), the answer to both questions 2 and 3 will be the same and the optimal solution is obtained as follows:

$$X_1 = 2, \quad X_2 = 3, \quad X_3 = 1, \quad X_4 = 4, \quad X_5 = 2$$

By increasing the value of Γ_i for production capacity, the uncertainty appears. For different Γ_i , we have provided the following answers.

Table 1. The amount of production capacity for Γ, P

	X_5	X_4	X_3	X_2	X_1	production capacity
$\hat{p}_i$	0.6	0.1	0.5	0.4	0.1	
$\Gamma_i = 0$	2	4	1	3	2	30
$\Gamma_i = 1$	2	3	1	4	2	29.7

According to the above table, it is clear that by increasing of Γ_i , different answers are obtained to the problem. Obviously, by increasing the uncertainty amount of parameters, the optimal solutions become more cautious. In other words, by increasing the level of uncertainty in parameters, the assignment mode of machines, their selection, and the type of machines placement will change. In the above table, it is clear that by increasing the uncertainty of parameters, the amount of using machine 2 increases and the amount of using machine 4 decreases. It is important to note that by increasing the uncertainty of parameters from zero to one, the entire system's production capacity is reduced from 30 to 29.7 and decreased optimal solution along with increasing uncertainty is seen.

The following table shows the reliability of each machine that has already been measured.

Table 2. The data is related to reliability and cost

System	1	2	3	4	5
Reliability	0.95	0.80	0.90	0.90	0.95
Cost	100	120	150	110	90

We solve $\Gamma_i = 1$:

Here, we are faced with cost constraint. The total cost and total budget that we have for spending in the system is equal to 300. Given this cost constraint and reliability of each machine, we should optimize the system's reliability. Therefore, by choosing the machine which increases the reliability of the entire system more than others, we add the machines to the system.

Considering the proposed heuristic function, the data of reliability table, cost of each of the devices, and the calculation of Δ_i , the following equation is obtained:

$$\Delta_2 > \Delta_3 > \Delta_5 > \Delta_1 > \Delta_4 \tag{13}$$

According to the obtained data , the highest value of Δ_i indicates a machine that should be added to the system before other machine. Therefore, considering the constraint of cost, reliability will have the highest increase.

Therefore, we add a unit of the second machine to the system. Since, some constraint of cost still remained, the total budget for the system was determined 300 and by adding only one unit from the machine 2, 120 units were spent, hence, we continue the above procedure to add another device.

The above procedure continues until the budget constraint will end and new machine cannot be added to the system.

5. Conclusion

In this paper, a two-objective problem was provided for a target function as reliability of a series-parallel system. The offered model in this paper includes an accurate production map for optimal placement of facilities in different production units by basic algorithms. In this paper, we assume that the production capacity for production units depends on the uncertainty in the first phase of the algorithm.

The second phase of this problem relates to the optimization of system reliability. The final answer to this question helps us obtain the best and most optimal method to get the best plan for target problem. It is also important to note that the heuristic method is offered for the second optimization function obtains optimal answer only when the problem has a constraint. The problems have more than one constraint cannot obtain optimal answer by using the provided method.

6. Appendix

Lingo code for gamma equals zero ($\Gamma = 0$)

```
MODEL:
! Robust PRODUCTION Optimization WITH BERTSIMAS APPROACH;
SETS:
        STATION /S1..S5 /: X,PR,PRHAT,LB,UB,Y,P,Z,GAMA;
        M /M1..M3/:B;
        TECHNICAL (M,STATION):A;
ENDSETS
! The objective function;
[OBJ] MAX = W;
! ROBUST Constraints;
@FOR(STATION( i):
        W - PR( i)*X( i) + Z(i)*GAMA(i) + P( i)<=0);
@FOR( STATION( i):
         Z(i)+p( i) >= PRHAT( i)*Y( i));
@FOR( STATION( i): X( i) <=
        Y( i));
@FOR( STATION( i): X( i) >=
        -Y( i));
! Lower and upper bounds ;
@FOR( STATION( i):
        X( i) <=UB(i));
@FOR( STATION( i):
         X( i) >= LB(i));
! INTEGER constraints;
@FOR( STATION( i): @GIN( X));
@FOR( M( i):
        @SUM( STATION( j): A(i,j)*X(j))<= B( i));
PRODUCTION=@MIN(STATION(i): PR(i)*X(i));
DATA:
        GAMA= 0     0     0     0     0;
        PRHAT= 0.1  0.4   0.5   0.1   0.6;
        PR=    20   10    50    10    30;
```

```
    LB=     1   1    1    1    1;
    UB=     3   12   4    6   3;
    B=30   40        70;
    A=2    1         1         4         3
      2    1         4         2         6
      3    2         5         7         3;
ENDDATA
END
```

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